Probability and Random Processes
EES 315

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Summary of Chapters 11-13

Chapter 6 vs. Chapter 11

Joint probability

$$P(\underbrace{A \cap B}_{\text{Joint event}})$$

Joint pmf

$$p_{X,Y}(x,y) = P[X = x, Y = y]$$

$$A = [X = x]$$

$$B = [Y = y]$$

Conditional probability

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$= \frac{P(B|A)P(A)}{P(B)}$$

Conditional pmf

$$p_{X|Y}(x \mid y) = \frac{p_{X,Y}(x,y)}{p_Y(y)}$$

$$= \frac{p_{Y|X}(y \mid x)\, p_X(x)}{p_Y(y)}$$

Events A and B are **independent**:

$$P(A \cap B) = P(A)P(B)$$

RVs X and Y are **independent**:

$$p_{X,Y}(x,y) = p_X(x)\, p_Y(y) \text{ for any } x \text{ and } y$$

i.i.d. RVs

- Two random variables X and Y said to be **identically distributed** if

  $p_X(c) = p_Y(c)$ for all c
- Two random variables X and Y are said to be **independent and identically distributed (i.i.d.)** if

  X and Y are both independent and identically distributed.
- Example: Roll a dice twice.
  - Let X be the result from the first roll.
  - Let Y be the result from the second roll.

Chapter 9 vs. Section 10.3

One Discrete RV	Two Discrete RVs
$\mathbb{E}X = \sum_x x p_X(x)$	$\mathbb{E}[g(X,Y)] = \sum_x \sum_y g(x,y) p_{X,Y}(x,y)$
$\mathbb{E}[g(X)] = \sum_x g(x) p_X(x)$	
$\mathbb{E}[X^2] = \sum_x x^2 p_X(x)$	$\mathbb{E}[XY] = \sum_x \sum_y xy p_{X,Y}(x,y)$ Correlation
$\operatorname{Var}[X] = \mathbb{E}\left[(X - \mathbb{E}X)^2\right] = \mathbb{E}[X^2] - (\mathbb{E}X)^2$ $\sigma_X = \sqrt{\operatorname{Var}[X]}$	Covariance $\operatorname{Cov}[X,Y] = \mathbb{E}[(X - \mathbb{E}X)(Y - \mathbb{E}Y)] = \mathbb{E}[XY] - (\mathbb{E}X)(\mathbb{E}Y)$ $\rho_{X,Y} = \frac{\operatorname{Cov}[X,Y]}{\sigma_X \sigma_Y}$ Correlation coefficient

Correlation

Actually, it's the "correlation coefficient"

- Correlation measures a specific kind of dependency.
  - Dependence = statistical relationship between two random variables (or two sets of data).
  - Correlation measures **"linear" relationship** between two random variables.

Extension to more than 2 RVs

- Consider n random variables X_1, X_2, and X_n
- **Joint pmf**:
  "and"
  $\boldsymbol{p_{X_1,X_2,\dots,X_n}(x_1,x_2,\dots,x_n)} = P[X_1 = x_1, X_2 = x_2, \dots, X_n = x_n]$
- They are said to be **identically distributed** if
  $p_{X_1}(c) = p_{X_2}(c) = \cdots = p_{X_n}(c)$ for all c
- They are said to be **independent** if
  $p_{X_1,X_2,\dots,X_n}(x_1,x_2,\dots,x_n) = p_{X_1}(x_1)p_{X_2}(x_2)\cdots p_{X_n}(x_n)$
- They are said to be **independent and identically distributed (i.i.d.)** if
  X_1, X_2, and X_n are both independent and identically distributed.

CH12: Limiting Theorems

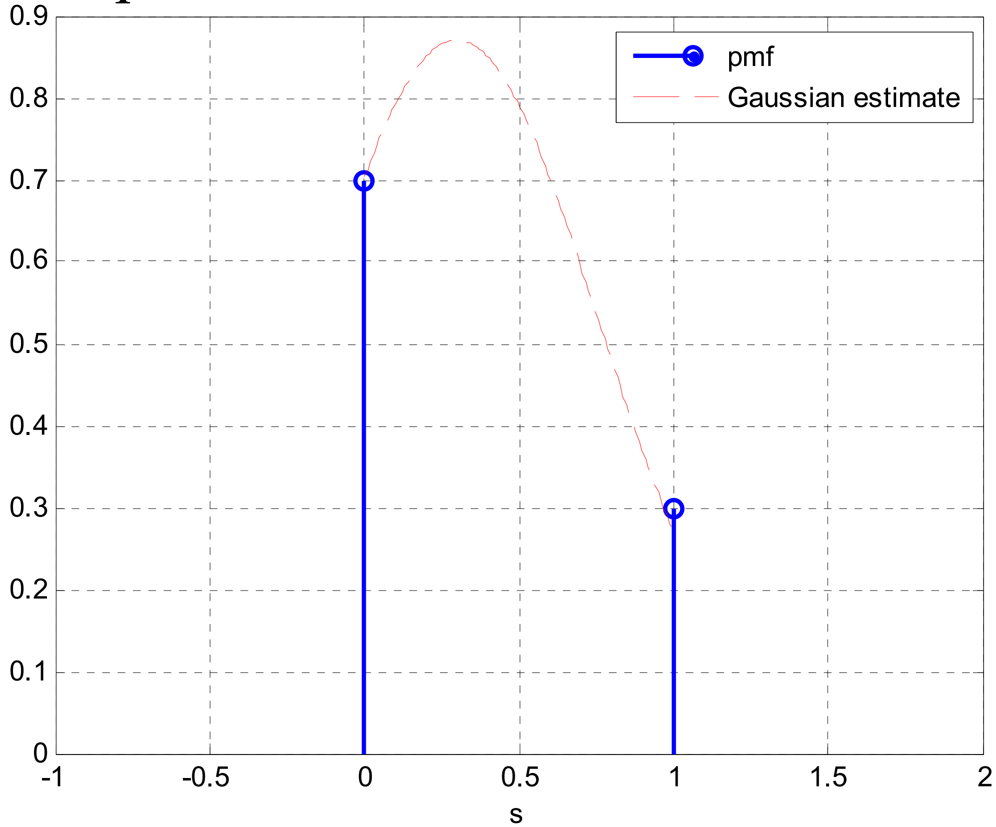
- Consider a sequence of **i.i.d. random variables** X_1, X_2, and X_n
  - Let $m = \mathbb{E}[X_1] = \mathbb{E}[X_2] = \cdots$
  - Let $\sigma^2 = \text{Var}[X_1] = \text{Var}[X_2] = \cdots$
- **Law of Large Numbers (LLN)**:
  The sample means
  $$M_n = \frac{1}{n}\sum_{k=1}^{n} X_k$$
  will converge to the shared expected value m as $n \to \infty$.
  - Corollary: relative frequency will converge to probability.
- **Central Limit Theorem (CLT)**:
  For n large enough, we can approximate their sum
  $$S_n = \sum_{k=1}^{n} X_k$$
  by a Gaussian random variable with the mean mn and variance $n\sigma^2$.

Sum of *n* Bernoulli RVs

- Let
  - $X_1, X_2, \ldots$ be i.i.d. Bernoulli(p).
  - $S_n = \sum_{k=1}^{n} X_k$.
- Let's try to plot the pmf of S_n.

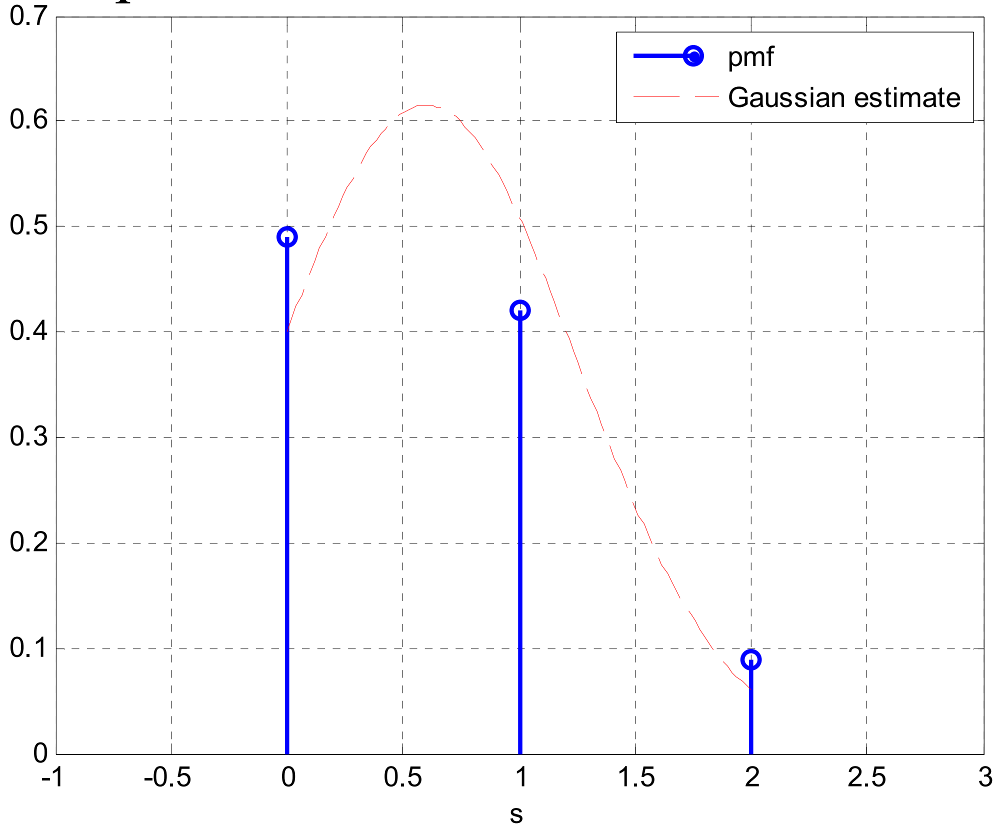
A Bernoulli RV

- Let
  - $X_1, X_2, \dots$ be i.i.d. Bernoulli(p).
  - $S_n = \sum_{k=1}^{n} X_k$.
- Let's try to plot the pmf of S_n.
- $n = 1$:



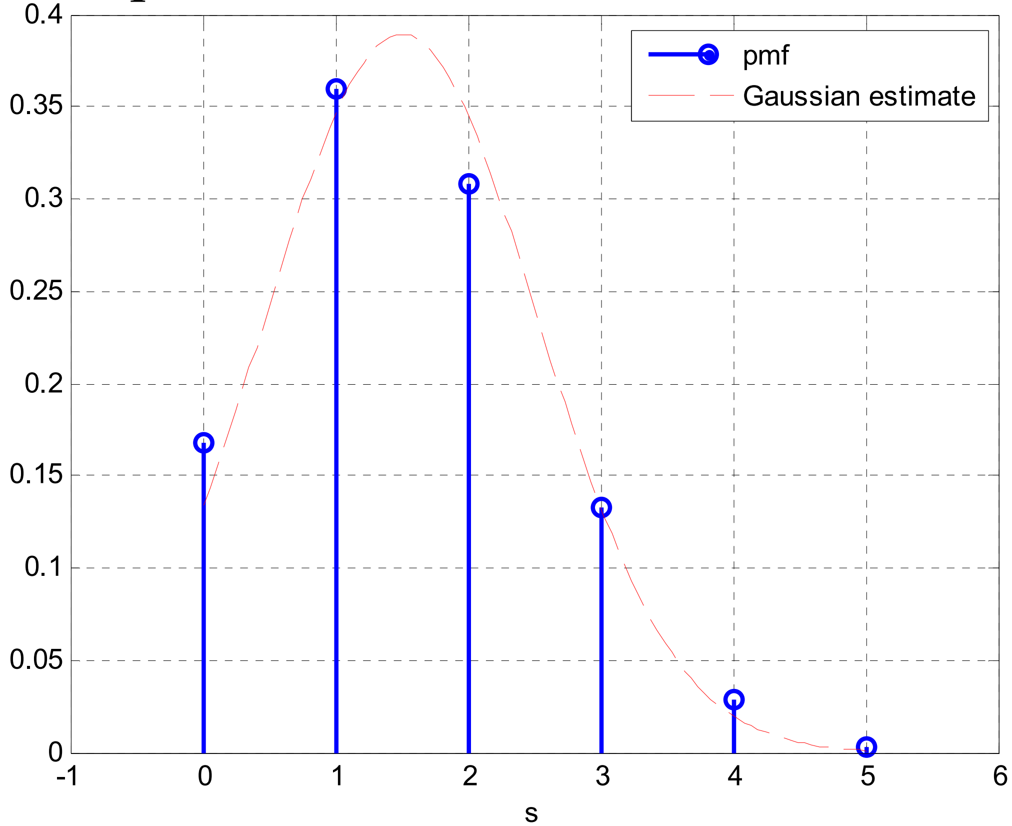
Sum of *two* Bernoulli RVs

- Let
  - $X_1, X_2, \ldots$ be i.i.d. Bernoulli(p).
  - $S_n = \sum_{k=1}^{n} X_k$.
- Let's try to plot the pmf of S_n.
- $n = 2$:



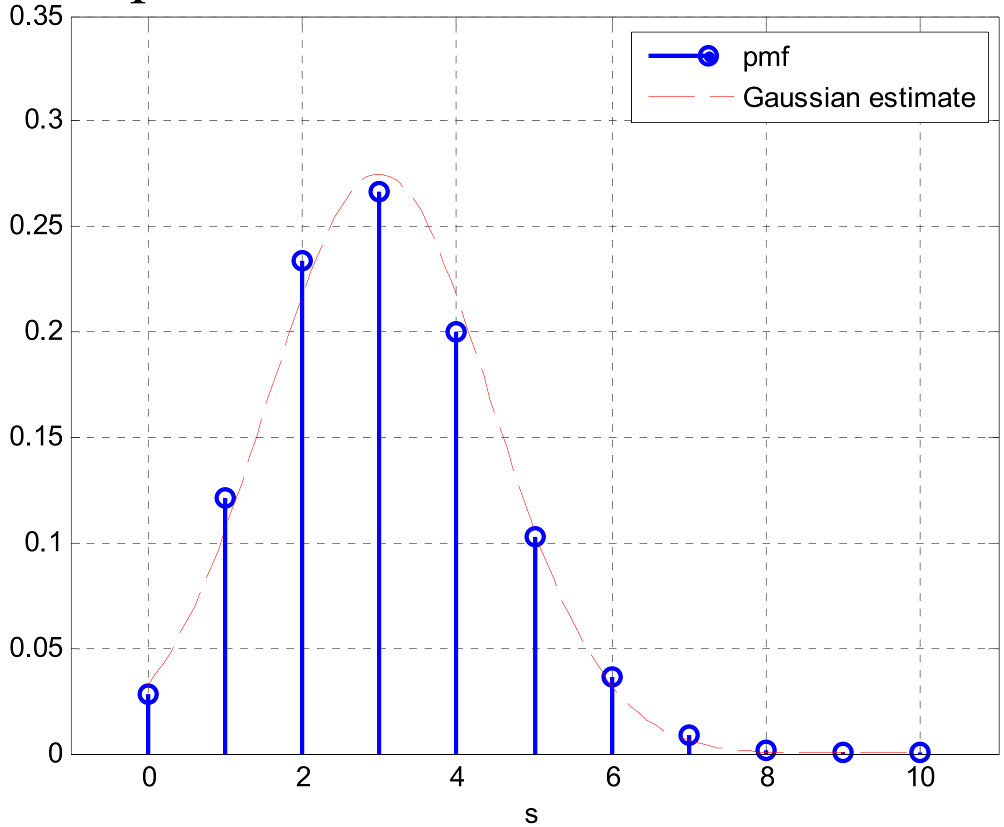
Sum of *five* Bernoulli RVs

- Let
  - $X_1, X_2, \ldots$ be i.i.d. Bernoulli(p).
  - $S_n = \sum_{k=1}^{n} X_k$.
- Let's try to plot the pmf of S_n.
- $n = 5$:



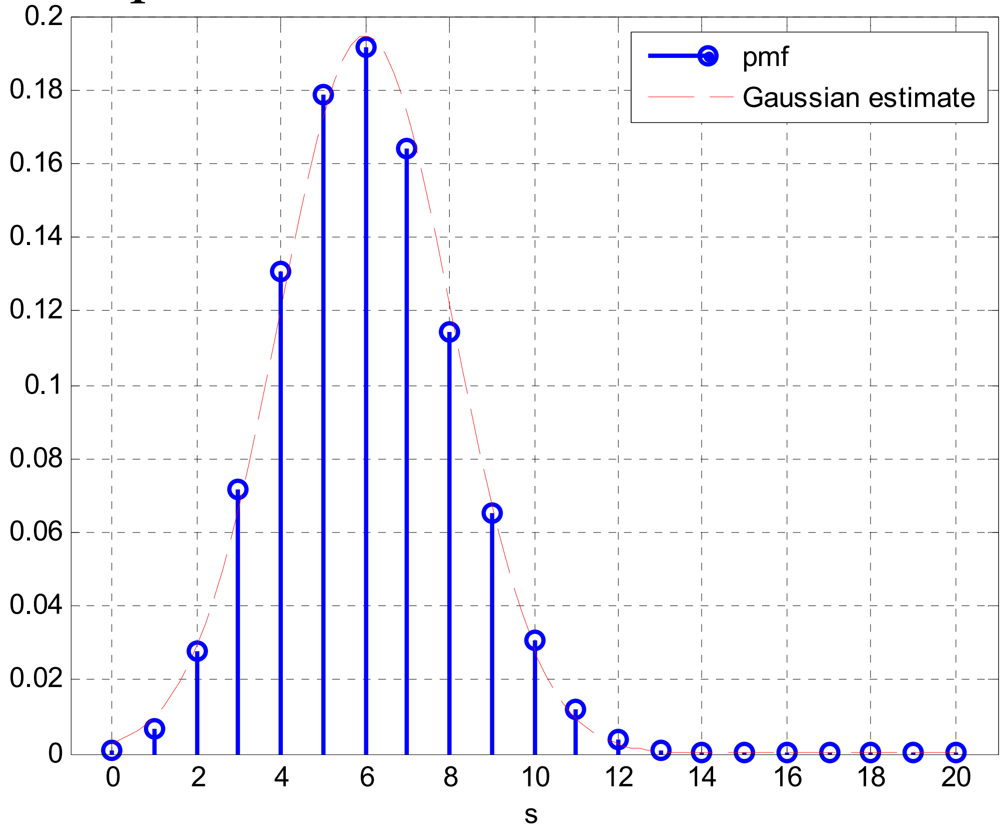
Sum of *ten* Bernoulli RVs

- Let
  - $X_1, X_2, \ldots$ be i.i.d. Bernoulli(p).
  - $S_n = \sum_{k=1}^{n} X_k$.
- Let's try to plot the pmf of S_n.
- $n = 10$:



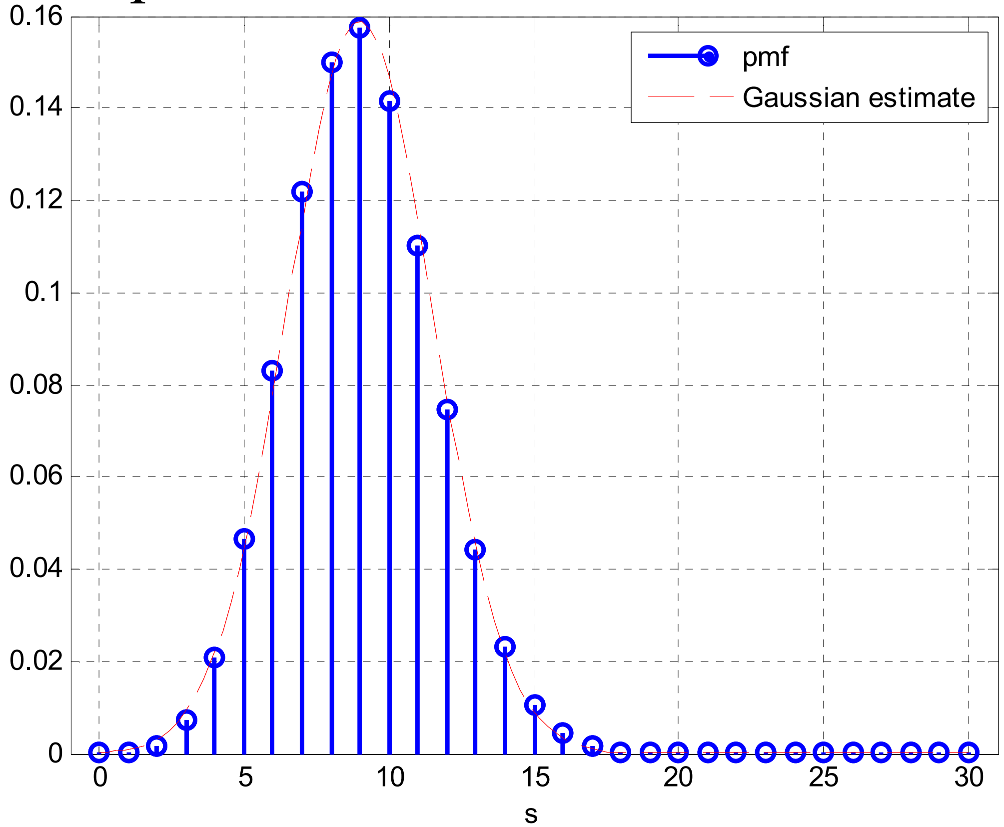
Sum of 20 Bernoulli RVs

- Let
  - $X_1, X_2, \ldots$ be i.i.d. Bernoulli(p).
  - $S_n = \sum_{k=1}^{n} X_k$.
- Let's try to plot the pmf of S_n.
- $n = 20$:



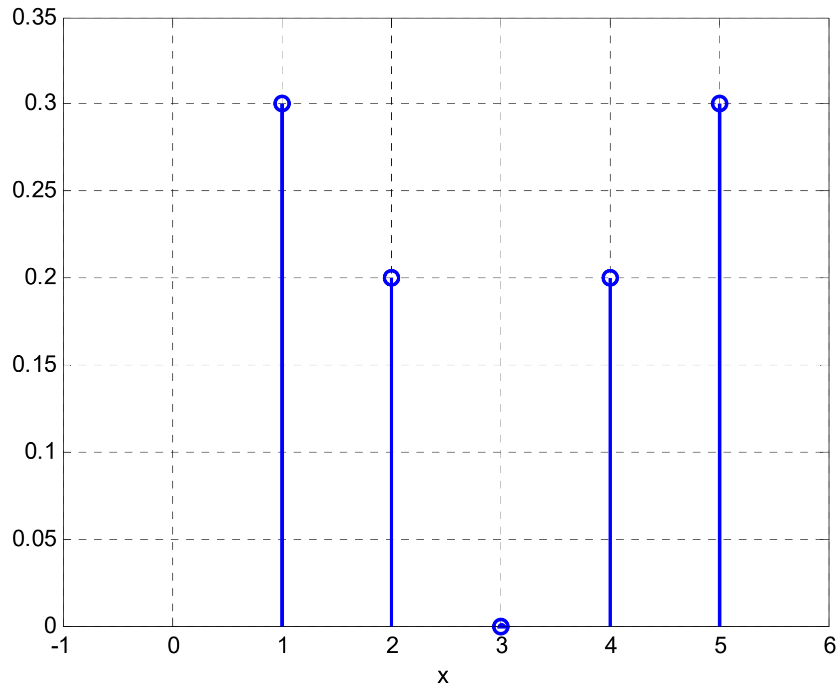
Sum of 30 Bernoulli RVs

- Let
  - $X_1, X_2, \ldots$ be i.i.d. Bernoulli(p).
  - $S_n = \sum_{k=1}^{n} X_k$.
- Let's try to plot the pmf of S_n.
- $n = 30$:



Sum of *n* "Arbitrary" Discrete RVs

- Let
  - $X_1, X_2, \ldots$ be i.i.d.
    - Here is their shared pmf:
  - $S_n = \sum_{k=1}^{n} X_k$.
- Let's try to plot the pmf of S_n.



Sum of *n* "Arbitrary" Discrete RVs

- Let
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    - Here is their shared pmf: →
  - $S_n = \sum_{k=1}^{n} X_k$.
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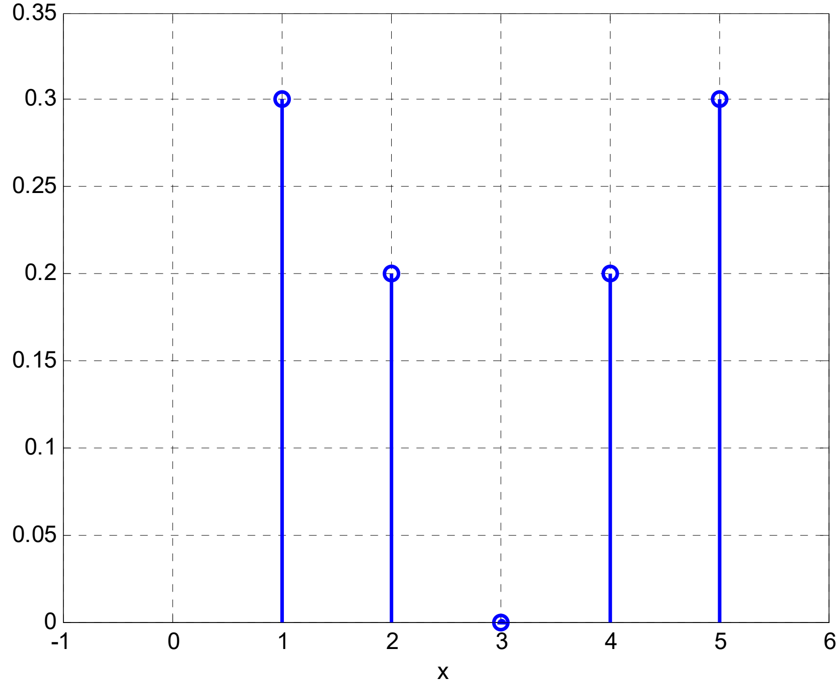

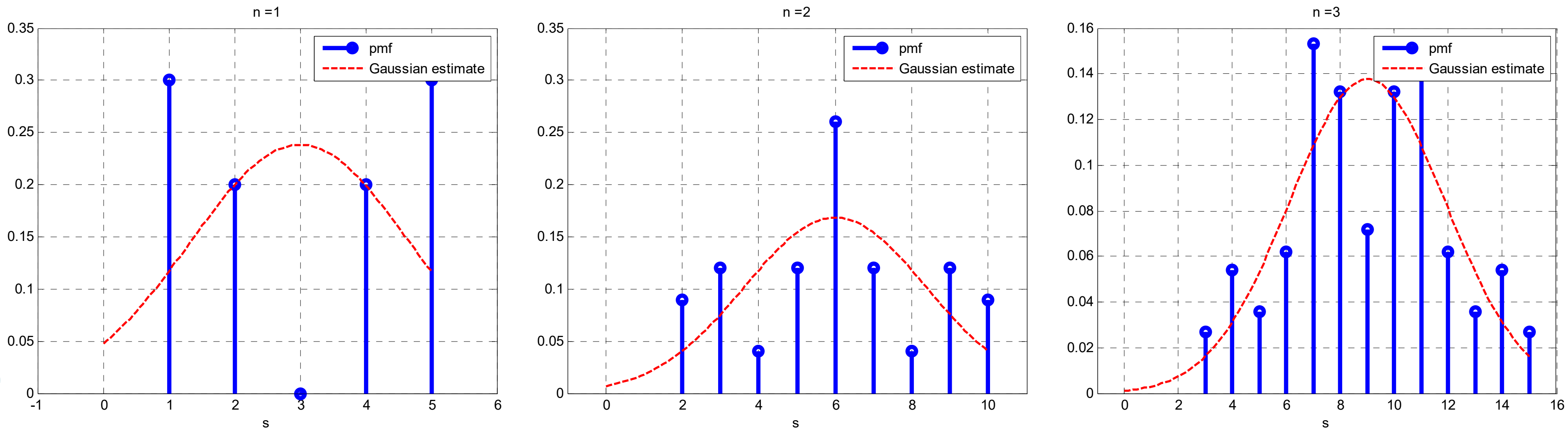

Sum of *n* "Arbitrary" Discrete RVs

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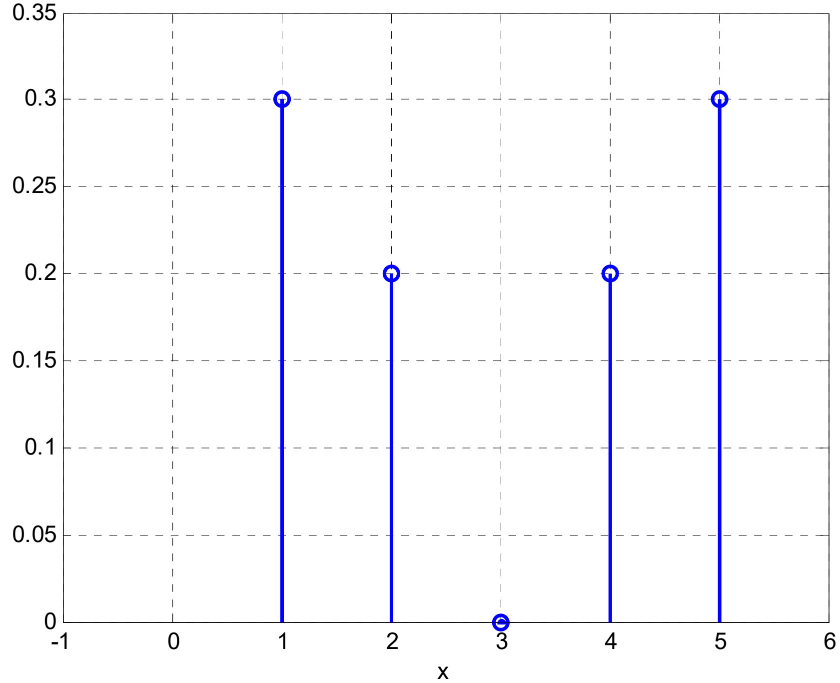

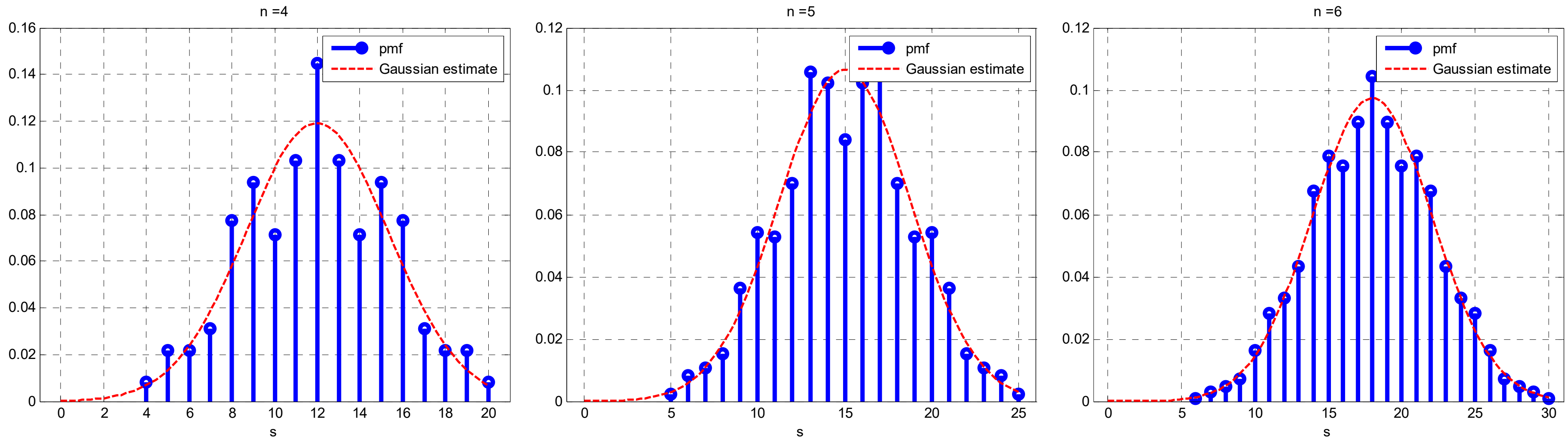

Sum of *n* "Arbitrary" Discrete RVs

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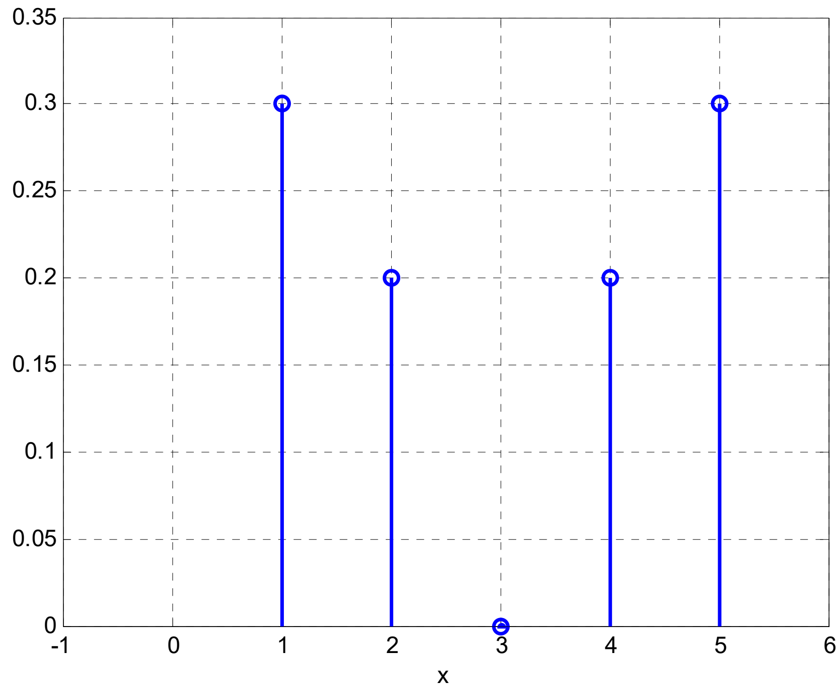
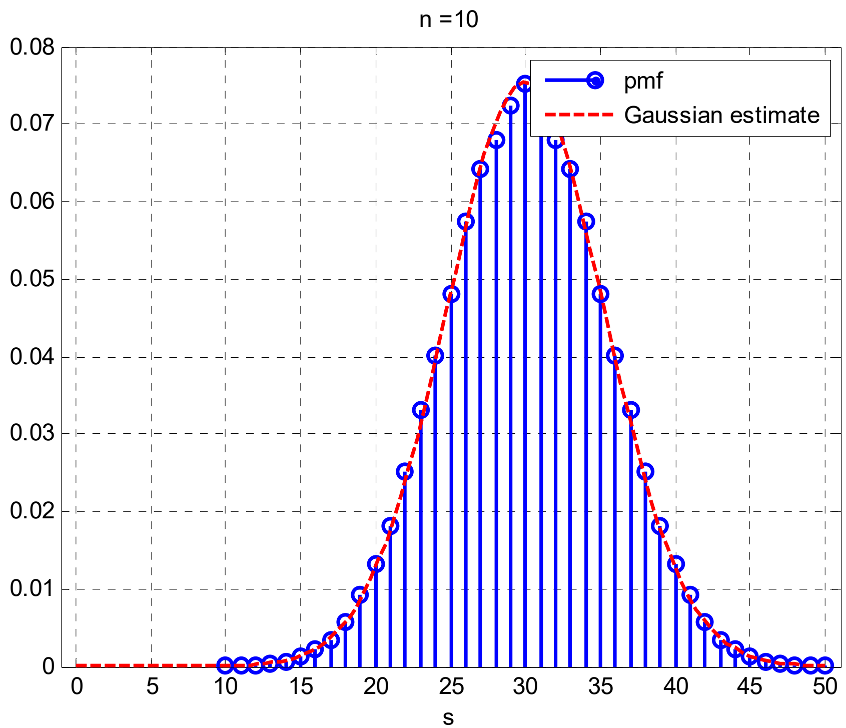
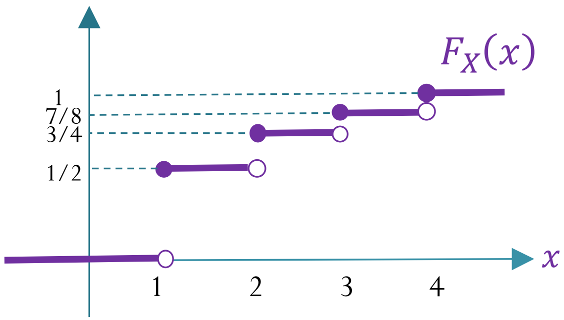




CH13: Three Types of RVs

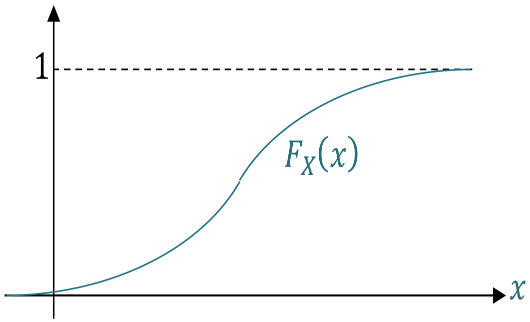
Discrete RV

cdf is a staircase function with jumps whose size at $x = c$ gives $P[X = c]$.



Continuous RV

cdf is a continuous function.



Mixed RV